A note on robust control of nonlinear systems with input unmodeled dynamics

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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

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A note on “Robust control of nonlinear systems with input unmodeled dynamics”

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Abstract: In this note, we point out alternative proofs, and alternative controllers, for the results of the paper “Robust control of nonlinear systems with input unmodeled dynamics”. In that paper, Krstić, Sun and Kokotović solve various stabilization problems in presence of input unmodeled dynamics. Their solution incorporates, in the controller, a signal, called normalizing signal, which captures the effect of the unmodeled dynamics. Here, we show that this signal is not needed. This result is obtained via an appropriate change of coordinates of the unmodeled dynamics and by applying the technique of propagating the ISS property through integrators, due to Jiang, Praly and Teel. The dimension of the state of our controllers is smaller than that of the controllers proposed by Krstić, Sun and Kokotović.

Key-words: nonlinear systems, global asymptotic stability, input-to-state stability (ISS), nonlinear small-gain

(Résumé : tsvp)

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Une note sur la commande robuste de systèmes nonlinéaires avec dynamiques non-modélisées sur l'entrée

Résumé : L'objet de cette note est de proposer, grâce à de nouvelles lois de commandes, de nouvelles preuves des résultats obtenus dans l'article "Robust control of nonlinear systems with input unmodeled dynamics", où Krstić, Sun et Kokotović résolvent un certain nombre de problèmes de stabilisation en présence de dynamiques non modélisées affectant les entrées. Leur solution fait apparaître, dans le contrôleur, un "signal normalisateur" qui détecte les effets des dynamiques non modélisées. Nous montrons ici que cela n'est pas nécessaire. La solution que nous proposons consiste, après un changement de coordonnées qui rend ceci possible, à appliquer la technique d'ajouts d'intégrateurs en propageant la propriété de stabilité entrée-état selon une technique due à Jiang, Praly et Teel. La taille de l'état des contrôleurs suggérés ici est inférieure à celle du contrôleur proposé par Krstić, Sun et Kokotović.

Mots-clé : systèmes non-linéaires, stabilité asymptotique globale, stabilité entrée-état, petits-gains non-linéaires

1 Introduction

In a recent series of the papers [1, 2, 3, 7, 9, 10], various versions of the small-gain theorem have been established for general nonlinear systems and have been used to tackle local and/or global stabilization problems of nonlinear systems in presence of state driven dynamic perturbations. More recently, Krstić, Sun and Kokotović [5] have stated the problem of input driven unmodeled dynamics. They have proposed a solution for the case where these dynamics are linear and the nominal system is basically of the feedback form. Both state and output feedback have been studied. But the main originality is in the incorporation of a so-called *dynamic nonlinear damping*, technique borrowed from the adaptive linear control field [6].

In this note, we take advantage of the linearity assumptions made in [5] to show that the input driven unmodeled dynamics can be embedded in state driven unmodeled dynamics for which the solution is already available as mentioned above. As a consequence, this extra dynamic nonlinear damping signal appears as not necessary (although it may be helpful for performance).

We refer the reader to [8, 3, 1] for basic definitions on input-to-state stability (ISS), input-to-state practical stability (ISpS) and other related facts.

2 Design tools

As in [5, Lemma 3.1], we consider the control system subject to input unmodeled dynamics ξ :

$$\left\{ \begin{array}{rcl} \dot{x} &=& f(x) + g(x)(wu + \mu c\xi) \ , \quad x \in I\!\!R \, , \ u \in I\!\!R \\ \dot{\xi} &=& A\xi + bu \ , \quad \xi \in I\!\!R^p \end{array} \right. \tag{1}$$

where f, g are smooth known functions, A, b and c are unknown matrices of appropriate dimensions, A is asymptotically stable, and w, μ are unknown constants.

We have :

Lemma 1 *Assume g satisfies :*

$$x \neq 0 \quad \Longrightarrow \quad g(x) \neq 0 \ . \tag{2}$$

Assume further that there exists a smooth function α such that $\dot{x} = f(x) + g(x)\alpha(x)$ is GAS at $x = 0$. Under these conditions, given a strictly positive real number δ_w, there exist a positive real number μ^* and a smooth function ϑ such that, for all μ in $(-\mu^*, \mu^*)$, the closed-loop system (1) with $u = \vartheta(x) + v$ is ISpS with v as input provided that $w \geq \delta_w$. Furthermore, if g is such that $g(x) \geq \delta_g > 0$ for all x in $I\!R$, we can render the closed-loop system ISS, and LES when $v = 0$.

Proof. Pick any ε_0 in $(0, \infty)$ and let φ be a smooth function such that $0 \leq \varphi(x) \leq 1$ for all $x \in I\!R$, $\varphi(x) = 1$ for x in $(-\infty, -\varepsilon_0] \cup [\varepsilon_0, \infty)$, and $\varphi(x)/g(x)$ is a C^1 function. Then introduce the following new variable :

$$\bar{\xi} \;=\; \mu\xi - \frac{\mu b}{w}\int_0^x \frac{\varphi(s)}{g(s)} ds \;. \tag{3}$$

Note that if $g(0) \neq 0$, we can take $\varphi \equiv 1$. In the new system of coordinates $(x, \bar{\xi})$, the system (1) is rewritten as :

$$\dot{x} \;=\; f(x) + g(x)\left(wu + c\bar{\xi} + \frac{\mu c b}{w}\int_0^x \frac{\varphi(s)}{g(s)} ds\right) \tag{4}$$

$$\begin{aligned}\dot{\bar{\xi}} \;&=\; \left(A - \mu\frac{\varphi(x)}{w}bc\right)\bar{\xi} + \left(A - \mu\frac{\varphi(x)}{w}bc\right)\frac{\mu b}{w}\int_0^x \frac{\varphi(s)}{g(s)} ds \\ &\quad - \frac{\mu b \varphi(x) f(x)}{w g(x)} \;+\; \mu b(1 - \varphi(x))u \;.\end{aligned} \tag{5}$$

Now to any fixed C^0 function $u(x)$, we can associate the positive real number :

$$d_u \;=\; \sup_{x \in I\!R} |b(1 - \varphi(x))u(x)| \;=\; \max_{|x| \leq \varepsilon_0} |b(1 - \varphi(x))u(x)| < +\infty \;. \tag{6}$$

This number is 0 if $g(0) \neq 0$. Also, we see that, given any (fixed) positive real number k_0, there exist positive real numbers μ^* and k such that, for all μ in $(-\mu^*, \mu^*)$, the $\bar{\xi}$-system (5) with input x and output $\left(\frac{c}{w}\bar{\xi} + \frac{\mu c b}{w^2}\int_0^x \frac{\varphi(s)}{g(s)} ds\right)$ is input-to-output practically stable (IOpS) (resp. IOS if $g(0) \neq 0$) [1, 3] with gain function $\gamma_{\bar{\xi}}$:

$$\gamma_{\bar{\xi}}(s) \;=\; k\,|\mu|\,\hat{\gamma}_{\bar{\xi}}(s) \qquad \forall\, \mu \in (-\mu^*, \mu^*) \;, \tag{7}$$

$$\hat{\gamma}_{\bar{\xi}}(s) \;=\; k_0\left(s + \max_{|x| \leq s}\left\{\left|\int_0^x \frac{\varphi(s)}{g(s)} ds\right| + \left|\frac{\varphi(x) f(x)}{g(x)}\right|\right\}\right) \;. \tag{8}$$

This means that this gain $\gamma_{\bar{\xi}}$ is known up to the multiplication by a real number and is linearly bounded on a neighborhood of 0. Note that k in (7) goes to 0 as k_0 goes to ∞.

With in mind the idea of applying the recent Small-Gain Theorem [3, Theorem 2.1] or [1, Théorème 1.9], we follow the technique of gain assignment proposed in [3, Corollary 2.2] or [1, Théorème 2.1] for the system :

$$\dot{x} \ = \ f(x) + g(x)w(u+d) \tag{9}$$

with u and d as input and x as state. [3, Corollary 2.2] or [1, Théorème 2.1] do not apply readily since $g(x)$ may vanish. Nevertheless, with (2), we see that the result can be easily extended to this case.

By hypothesis, there exist a function γ_3 of class K_∞ and a smooth positive definite and proper function V such that :

$$\frac{\partial V}{\partial x}(x)[f(x) + g(x)\alpha(x)] \ \leq \ -\gamma_3(|x|) \quad \forall x \in I\!\!R \ . \tag{10}$$

Taking the time derivative of V along the solutions of (9), we have :

$$\dot{V} \ \leq \ -\gamma_3(|x|) + \frac{\partial V}{\partial x}(x)g(x)\left[w(u+d) - \alpha(x)\right] \ . \tag{11}$$

Following the same lines as in [7, Lemma A.1] (also see [1, Lemme C.5]), we choose a smooth function γ_x which is of class K_∞, linear on a neighborhood of 0 such that :

$$\hat{\gamma}_{\bar{\xi}}(2s) \leq \gamma_x^{-1}(s) \quad \forall\, s \geq 0 \ . \tag{12}$$

The inverse function γ_x^{-1} of γ_x is also smooth, of class K_∞ and linear around zero.

Because of (2) and the fact that $\frac{\partial V}{\partial x}(x) = 0$ iff $x = 0$ (see (10)), we can divide our study into the following three cases :

Case (a) : The sign of $\frac{\partial V}{\partial x}(x)g(x)$ is constant, i.e. $|\frac{\partial V}{\partial x}g| \equiv \frac{\partial V}{\partial x}g$ or $\equiv -\frac{\partial V}{\partial x}g$.
Case (b) : $\frac{\partial V}{\partial x}(x)g(x)x > 0$ for all $x \neq 0$.
Case (c) : $\frac{\partial V}{\partial x}(x)g(x)x < 0$ for all $x \neq 0$.

In Case (a), by choosing the following smooth control law :

$$\begin{aligned} \vartheta(x) &= -\frac{1}{4\varepsilon_1\delta_w}\alpha(x)^2\frac{\partial V}{\partial x}(x)g(x) \\ &\quad - 2(1+\varepsilon_2)\mathrm{sgn}\left(\frac{\partial V}{\partial x}(x)g(x)\right)\left[\gamma_x^{-1}(\sqrt{x^2+\varepsilon_1}) - \gamma_x^{-1}(\sqrt{\varepsilon_1})\right] \end{aligned} \tag{13}$$

with ε_1 and ε_2 two strictly positive real numbers. Then (11) with $u = \vartheta(x) + v$ gives :

$$\begin{aligned} \dot{V} \;\leq\; & -\gamma_3(|x|) \;+\; \varepsilon_1 \\ & + w\left|\frac{\partial V}{\partial x}(x)g(x)\right|\left[|d+v| - 2(1+\varepsilon_2)\left(\gamma_x^{-1}(\sqrt{x^2+\varepsilon_1}) - \gamma_x^{-1}(\sqrt{\varepsilon_1})\right)\right] \end{aligned} \tag{14}$$

According to Sontag's ISS algorithm [8], from (14), it follows that there exist a function β_x of class KL and two functions γ_v, d_x of class K such that, associated to any measurable essentially bounded controls $\bar{\xi}$, $v : I\!R_+ \to I\!R^p$, the solution $x(t)$ exists for all $t \geq 0$ and satisfies :

$$|x(t)| \;\leq\; \beta_x(|x(0)|, t) + \gamma_x(\|v_t\|) + \gamma_x(\|d_t\|) + d_x(\varepsilon_1) \tag{15}$$

Namely, the x-system (4) with $u = \vartheta(x) + v$ is ISpS with (v, d) as input and γ_x as gain.

In Case (b), the same conclusion holds. Indeed, in this case, we choose the following smooth control law in place of (13) :

$$\vartheta(x) \;=\; -\frac{1}{4\varepsilon_1\delta_w}\alpha(x)^2\frac{\partial V}{\partial x}(x)g(x) \;-\; 2(1+\varepsilon_2)\gamma_x^{-1}(|x|)\mathrm{sgn}(x)\ . \tag{16}$$

For the case (c), the procedure is similar.

Therefore, thanks to (12), the nonlinear small-gain theorem [3, Theorem 2.1], or [1, Théorème 1.9] allows us to conclude the first point of Lemma 1 if :

$$\mu^*\, k \;\leq\; 1\ . \tag{17}$$

The second point is proved as in [3, Corollary 2.2] or [1, Théorème 2.1] just by taking advantage of the local exponential stability. □

Next as in [5, Lemma 3.2], for the higher-dimensional system with input unmodeled dynamics ξ :

$$\left\{\begin{array}{rcl} \dot{X} & = & F(X) + G(X)x \ , \quad X \in I\!R^{n-1} \\ \dot{x} & = & f(X,x) + g(X,x)(wu + \mu c\xi) \ , \quad x \in I\!R \\ \dot{\xi} & = & A\xi + bu \ , \quad \xi \in I\!R^p \end{array}\right. \tag{18}$$

with the same notations A, b, c, w, μ as in (1), we have :

Lemma 2 *Assume g satisfies :*

$$g(X,x) \geq \delta_g > 0 \quad \text{for all } (X,x) \text{ and some known } \delta_g \ . \tag{19}$$

Assume further that there exists a smooth function α which is zero at zero such that $\dot{X} = F(X) + G(X)\alpha(X)$ is GAS at $x = 0$. Under these conditions, given a strictly positive real number δ_w, there exist a positive constant μ^* and two smooth functions K, ϑ which are zero at zero such that, for all μ in $(-\mu^*, \mu^*)$, the system (18) in closed-loop with $u = \vartheta(x - K(X)) + v$ is ISpS with v as input provided that $w \geq \delta_w$. Moreover, if $\dot{X} = F(X) + G(X)\alpha(X)$ is LES at $X = 0$, then we can render the closed-loop system ISS, and LES when $v = 0$.

Proof. In the light of [8, Theorem 1], there exists a smooth scalar function K, $K(0) = 0$, so that $\dot{X} = F(X) + G(X)(K(X) + u)$ is ISS with u as input with a known gain function. Introduce the following new variables :

$$\begin{aligned} \overline{x} &= x - K(X) && (20) \\ \bar{\xi} &= \mu\xi - \frac{\mu b}{w}\int_0^x \frac{ds}{g(X,s)} \ . && (21) \end{aligned}$$

In coordinates $(X, \overline{x}, \bar{\xi})$, the system (18) is rewritten as :

$$\begin{aligned} \dot{X} &= F(X) + G(X)(K(X) + \overline{x}) && (22) \\ \dot{\overline{x}} &= \overline{f}(X,\overline{x}) + \overline{g}(X,\overline{x})\left(wu + c\bar{\xi} + \frac{\mu cb}{w}\int_0^{K(X)+\overline{x}} \frac{ds}{g(X,s)}\right) && (23) \\ \dot{\bar{\xi}} &= \left(A - \mu\frac{bc}{w}\right)\bar{\xi} + \left(A - \mu\frac{bc}{w}\right)\frac{\mu b}{w}\int_0^{K(X)+\overline{x}} \frac{ds}{g(X,s)} + \frac{\mu b}{w}\overline{\Delta}(X,\overline{x}) && (24) \end{aligned}$$

where :

$$\overline{f}(X,\overline{x}) = f(X,K(X)+\overline{x}) + \frac{\partial K}{\partial X}(X)\left[F(X)+G(X)(K(X)+\overline{x})\right] \quad (25)$$

$$\overline{g}(X,\overline{x}) = g(X,K(X)+\overline{x}) \quad (26)$$

$$\overline{\Delta}(X,\overline{x}) = -\frac{f(X,\overline{x}+K(X))}{g(X,\overline{x}+K(X))} + \int_0^{\overline{x}+K(X)} \frac{\frac{\partial g}{\partial X}(X,s)}{g(X,s)^2}\left[F(X)+G(X)(K(X)+\overline{x})\right] ds \quad (27)$$

We remark that the X-subsystem with $\overline{x}$ as input and $\frac{\mu cb}{w^2}\int_0^{K(X)+\overline{x}}\frac{ds}{g(X,s)}$ as output is IOS with a gain function which is known up to the multiplication by a real number which goes to 0 as μ goes to 0. Also, with our choice of $K(x)$, there exists a positive constant μ^* such that, for all μ in $(-\mu^*,\mu^*)$, the $\bar{\xi}$-system (24) is IOS with $\overline{x}$ as input and $\frac{c}{w}\bar{\xi}$ as output. Moreover, as in the proof of Lemma 1, the gain function is, as above, known up to the multiplication by a real number which goes to 0 as μ goes to 0. So, the $(X,\bar{\xi})$-system is IOS with $\overline{x}$ as input, $\frac{c}{w}\bar{\xi}+\frac{\mu cb}{w^2}\int_0^{K(X)+\overline{x}}\frac{ds}{g(X,s)}$ as output and a gain function which is known up to the multiplication by a real number which goes to 0 as μ goes to 0. Moreover this gain function is linearly bounded on a neighborhood of 0 whenever $\dot{X}=F(X)+G(X)K(X)$ is LES at $X=0$. The latter is guaranteed if $\dot{X}=F(X)+G(X)\alpha(X)$ is LES at $X=0$ (see [8, Proof of Theorem 1]). Therefore the proof is concluded as for Lemma 1. □

3 Main results

As in [5, Section 4], we first consider the so-called feedback linearizable system :

$$\left\{\begin{array}{rcl} \dot{x}_i &=& x_{i+1}\ , \quad 1\le i\le n-1 \\ \dot{x}_n &=& f(x)+g(x)(wu+\mu c\xi) \\ \dot{\xi} &=& A\xi+bu\ , \quad \xi\in I\!R^p \end{array}\right. \quad (28)$$

with A, b, c, w, μ as in (1), $w\ge\delta_w>0$, $f(0)=0$ and $g(x)\ge\delta_g>0$ for all $x=(x_1,\dots,x_n)$ in $I\!R^n$.

Noticing that there exist appropriate constants k_i's such that $u = \sum_{i=1}^{n-1} k_i x_i$ globally exponentially stabilizes

$$\left\{ \begin{array}{rcl} \dot{x}_i & = & x_{i+1} \ , \quad 1 \leq i \leq n-2 \\ \dot{x}_{n-1} & = & u \ , \end{array} \right. \tag{29}$$

a direct application of Lemma 2 gives :

Theorem 1 *We can design a smooth function ϑ, which is zero at zero, such that for some positive constant μ^* and all μ in $(-\mu^*, \mu^*)$, system (28) in closed-loop with $u = \vartheta(x_n - \sum_{i=1}^{n-1} k_i x_i)$ is GAS and LES at $(x, \xi) = (0, 0)$.*

Remark 1 In sharp contrast to [5, Theorem 4.1], we do not use dynamic feedback controls. Also, our control law exploits only the information of the scalar "output" $x_n - \sum_{i=1}^{n-1} k_i x_i$.

Then we study the strict-feedback systems with multiple "input" unmodeled dynamics ξ_i as in [5, Section 5] :

$$\left\{ \begin{array}{rcl} \dot{x}_i & = & w_i x_{i+1} + \mu c_i \xi_i + f_i(x_1, \ldots, x_i) \\ \dot{\xi}_i & = & A_i \xi_i + b_i x_{i+1} \ , \quad \xi_i \in I\!R^{p_i} \ , \ i \in \{1, \ldots, n-1\} \\ \dot{x}_n & = & w_n u + \mu c_n \xi_n + f_n(x_1, \ldots, x_n) \\ \dot{\xi}_n & = & A_n \xi_n + b_n u \ , \quad \xi_n \in I\!R^{p_n} \end{array} \right. \tag{30}$$

where f_i's are smooth and zero at zero, A_i's, b_i's and c_i's are unknown matrices of appropriate dimensions, A_i's are asymptotically stable, and w_i's, μ are unknown constant. Suppose that for i in $\{1, \ldots, n\}$, $0 < \delta_{w1} \leq w_i \leq \delta_{w2}$ for two known constants δ_{w1}, δ_{w2}.

For this system (30), like in the proof of Lemma 1, we introduce the following new variables :

$$\bar{\xi}_i \ = \ \mu \xi_i - \frac{\mu b_i}{w_i} x_i \ , \quad \forall \, i \in \{1, \ldots, n\} \ . \tag{31}$$

In the new system of coordinates $(x, \bar{\xi})$, system (30) is rewritten as :

$$\left\{ \begin{array}{rcl} \dot{x}_i & = & w_i x_{i+1} + c_i \bar{\xi}_i + \dfrac{\mu c_i b_i}{w_i} x_i + f_i(x_1, \dots, x_i) \\ \dot{\bar{\xi}}_i & = & (A_i - \mu \dfrac{b_i c_i}{w_i}) \bar{\xi}_i + (A_i - \mu \dfrac{b_i c_i}{w_i}) \dfrac{\mu b_i}{w_i} x_i - \dfrac{\mu b_i}{w_i} f_i(x_1, \dots, x_i) \\ & & i \in \{1, \dots, n\} \end{array} \right. \tag{32}$$

with $x_{n+1} = u$. Note that, for sufficiently small real numbers μ, each $\bar{\xi}_i$-subsystem of (32) is ISS with $(x_1, \dots, x_i)$ as input, and GES when $x_1 = \ldots = x_i = 0$. Hence system (32) differs from system (30) by the fact that (32) becomes a system with ISS unmeasured dynamics generated by the state.

Now we will apply the technique of propagating the ISS property through integrators proposed in [3, Section 4] (i.e. [3, Corollary 2.2, Proposition 3.2 & Fact 1]) to design smooth stabilizing controls for (32). For this, we first rewrite each x_i-subsystem of (32) as :

$$\dot{x}_i \;=\; w_i \left(x_{i+1} + \frac{c_i}{w_i} \bar{\xi}_i + \frac{\mu c_i b_i}{w_i^2} x_i + \frac{1}{w_i} f_i(x_1, \dots, x_i) \right) . \tag{33}$$

Then we remark that there exists a positive real number k_1 such that, for sufficiently small μ, the $\bar{\xi}_1$-system is input-to-output stable (IOS) with output

$$\frac{c_1}{w_1} \bar{\xi}_1 + \frac{\mu c_1 b_1}{w_1^2} x_1 + \frac{1}{w_1} f_1(x_1)$$

and gain function $\gamma_{\bar{\xi}_1}$:

$$\gamma_{\bar{\xi}_1}(s) \;=\; s + \frac{1}{\delta_{w1}} \max_{|x_1| \le s} |f_1(x_1)| + k_1 |\mu| \hat{\gamma}_{\bar{\xi}_1}(s) \tag{34}$$

$$\hat{\gamma}_{\bar{\xi}_1}(s) \;=\; \max_{|x_1| \le s} \{ |x_1| + |f_1(x_1)| \} . \tag{35}$$

This means that this gain function $\gamma_{\bar{\xi}_1}$ is known up to the multiplication by a real number and is linearly bounded on a neighborhood of zero. So given any positive real number k_{01}, there exists a positive real number μ^* such that $\gamma_{\bar{\xi}_1}$

is bounded by a known $\mathcal{K}_\infty$-function which is linearly bounded near zero, i.e. for all μ in $(-\mu^*, \mu^*)$ and all s in $I\!R_+$,

$$\gamma_{\bar{\xi}_1}(s) \leq s + \frac{1}{\delta_{w1}} \max_{|x_1| \leq s} |f_1(x_1)| + k_{01}\hat{\gamma}_{\bar{\xi}_1}(s) \ . \tag{36}$$

These properties will be used in the first step of the recursive design procedure [3, Section 4]. Similar properties can be established in the next steps.

Consequently, by applying *mutatis mutandis* the analysis developed in [3, Section 4] to system (32) (also see [1, Section 2.2.2]), we establish the following :

Theorem 2 *There exist a positive constant μ^* and a smooth partial state feedback $\vartheta(x_1, \ldots, x_n)$ such that, for all μ in $(-\mu^*, \mu^*)$, the null solution of the system (30) in closed-loop with $u = \vartheta(x_1, \ldots, x_n)$ is GAS.*

Next result concerns the perturbed output feedback system which has been studied in [5, Section 6] :

$$\begin{array}{rcl} \dot{x} & = & Fx + \phi(y) + g[(1+\mu q)\sigma(y)u + \mu c\xi] \ , \quad x \in I\!R^n \\ \dot{\xi} & = & A\xi + b\sigma(y)u \ , \quad \xi \in I\!R^p \\ y & = & x_1 \end{array} \tag{37}$$

$$F = \begin{pmatrix} 0 & \\ \vdots & I_{n-1} \\ 0 & \cdots \ 0 \end{pmatrix} \ , \quad g = (0, \ldots, 0, g_m, \ldots, g_0)^T \ , \ g_m \neq 0 \tag{38}$$

where q is a constant and A, b, c, μ are as in (1), ϕ is smooth and zero at zero, $\sigma(y) \neq 0$ for all y in $I\!R$. Assume that $g_m s^m + \cdots + g_1 s + g_0$ is a Hurwitz polynomial.

Our goal is to find a dynamic output-feedback $u = \vartheta(y, \mathcal{X})$, $\dot{\mathcal{X}} = \varpi(y, \mathcal{X})$ which globally asymptotically stabilizes (37) at $(x, \xi) = (0, 0)$.

For the purpose of getting such a stabilizing output-feedback controller for (37), inspired by the similarity transformation utilized in for example [4, eq. (3.16)], we intend to find an appropriate *linear* change of coordinates $(X, z, \bar{\xi}) = M(x, \xi)$ which transforms (37) into a special feedback form (cf.

eqs. (43), (62)) where all the dynamic perturbations are generated by the output y. Then using a separately designed partial-state observer as in [7], we are left with a special set of strict-feedback systems with ISS inverse dynamics ([3, Sec. 4]). So the problem is solved by applying our small-gain methods.

We first consider the case when $n-m=1$. It is easy to treat. Indeed, in this case, by choosing the following change of coordinates :

$$
\begin{aligned}
x_1 &= x_1 && (39)\\
z &= (H_0 e_1, I_{n-1})x && (40)\\
\bar{\xi} &= \mu\xi - \frac{\mu b}{g_{n-1}(1+\mu q)}x_1 && (41)
\end{aligned}
$$

where $e_1 = (1, 0\ldots, 0)^T$ in $\mathbb{R}^{n-1}$ and H_0 is an asymptotically stable matrix given by :

$$
H_0 = \begin{pmatrix} -g_{n-2}/g_{n-1} & \\ \vdots & I_{n-2} \\ -g_0/g_{n-1} & 0\cdots 0 \end{pmatrix}, \tag{42}
$$

we bring system (37) into :

$$
\begin{aligned}
\dot{x}_1 &= g_{n-1}(1+\mu q)\sigma(x_1)u + g_{n-1}c\bar{\xi} + (\frac{\mu cb}{1+\mu q} + \frac{g_{n-2}}{g_{n-1}})x_1 + B_0 z + \phi_1(x_1)\\
\dot{z} &= H_0 z + G_0(x_1)\\
\dot{\bar{\xi}} &= (A - \frac{\mu bc}{1+\mu q})\bar{\xi} + (A - \frac{\mu bc}{1+\mu q})\frac{\mu b x_1}{g_{n-1}(1+\mu q)} && (43)\\
&\quad - \frac{\mu b(B_0 z + \phi_1(x_1) + \frac{g_{n-2}}{g_{n-1}}x_1)}{g_{n-1}(1+\mu q)}
\end{aligned}
$$

where B_0 is a constant row vector and G_0, with $G_0(0)=0$, is the smooth function defined by :

$$
G_0(y) = -H_0^2 e_1 y + H_0 e_1 \phi_1(y) + (\phi_2(y), \ldots, \phi_n(y))^T . \tag{44}
$$

Note that, letting $\overline{z} = (z^T, \bar{\xi}^T)^T$, for sufficiently small μ, the obtained $\overline{z}$-system is ISS with x_1 as input (see [3, Proposition 3.2]), and GES when $x_1 = 0$. As a consequence there exists a positive real number μ^* such that, for all μ in $(-\mu^*, \mu^*)$, the $\overline{z}$-system is IOS with input x_1 and output :

$$\frac{c}{1+\mu q}\bar{\xi} + \left(\frac{\mu cb}{g_{n-1}(1+\mu q)^2} + \frac{g_{n-2}}{g_{n-1}^2(1+\mu q)}\right) x_1 + \frac{B_0 z}{g_{n-1}(1+\mu q)} + \frac{\phi_1(x_1)}{g_{n-1}(1+\mu q)}$$

and gain function $\gamma_{\overline{z}}$. In addition, this function $\gamma_{\overline{z}}$ is known up to the multiplication by an uncertain real number and is linearly bounded near zero (cf. (34)).

Then we can apply [3, Corollary 2.2] or [1, Théorème 2.1] to system (43) as in the proof of Lemma 1. Therefore, we arrive at (explicitly) designing a smooth static output-feedback $u = \vartheta(x_1)$ which globally asymptotically stabilizes system (43) and then system (37) in the case when $n - m = 1$.

In the sequel, we concentrate on the case where $n - m > 1$.

Inspired by the similarity transformation which was used for instance in [4, eq. (3.16)], we introduce the following change of coordinates :

$$\begin{array}{rcl} X_1 & = & x_1 \\ X_i & = & x_i - \displaystyle\sum_{j=1}^{i-1} l_{j,i} x_j \ , \quad 2 \le i \le n-m \\ z & = & \displaystyle\sum_{i=1}^{n-m} H_1^{n-m-i+1} e_1 x_i + (x_{n-m+1}, \dots, x_n)^T \\ \bar{\xi} & = & \mu\xi - \displaystyle\sum_{i=1}^{n-m} r_i x_i \end{array} \tag{45}$$

where e_1 stands again for $(1, 0 \dots, 0)^T$ but in $I\!R^m$, the $l_{j,i}$'s are real numbers and r_i's are vectors in $I\!R^p$ which are to be precised later, and H_1 is an asymptotically stable matrix given by :

$$H_1 = \begin{pmatrix} -g_{m-1}/g_m & \\ \vdots & I_{m-1} \\ -g_0/g_m & 0 \cdots 0 \end{pmatrix} . \tag{46}$$

From (45), there exist a constant row vector B_1 and real numbers d_i ($1 \le i \le n-m$) such that

$$x_{n-m+1} = \sum_{i=1}^{n-m} d_i x_i + B_1 z \ . \tag{47}$$

By choosing the r_i's in (45) as follows :

$$r_{n-m} = \frac{\mu b}{g_m(1+\mu q)} \tag{48}$$

$$r_i = (A - \frac{\mu bc}{1+\mu q})r_{i+1} - r_{n-m}d_{i+1} \quad \forall 1 \le i \le n-m-1 \;, \tag{49}$$

in view of (37) and (45), direct computation gives :

$$\begin{aligned} \dot{z} &= H_1 z \;+\; G_1(y) \\ \dot{\bar{\xi}} &= (A - \frac{\mu bc}{1+\mu q})\bar{\xi} \;+\; G_2(y) - \frac{\mu b}{g_m(1+\mu q)}B_1 z \end{aligned} \tag{50}$$

where G_1, G_2 are smooth functions defined by :

$$\begin{aligned} G_1(y) &= -H_1^{n-m+1}e_1 y + \sum_{i=1}^{n-m} H_1^{n-m-i+1}e_1\phi_i(y) + (\phi_{n-m+1}(y), \ldots, \phi_n(y))^T \\ G_2(y) &= \left[(A - \frac{\mu bc}{1+\mu q})r_1 - r_{n-m}d_1\right] y - \sum_{i=1}^{n-m} r_i\phi_i(y) \;. \end{aligned}$$

Note that, according to [3, Proposition 3.2], for sufficiently small μ, the z and $\bar{\xi}$ systems are ISS with y as input and a linear gain for small y.

With (47), the time derivative of X_{n-m} along the solutions of (37) is :

$$\begin{aligned} \dot{X}_{n-m} &= g_m(1+\mu q)\sigma(y)u + d_1 y + \phi_{n-m}(y) - \sum_{j=1}^{n-m-1} l_{j,(n-m)}\phi_j(y) + g_m c r_1 y \\ &\quad + g_m c\bar{\xi} + B_1 z + \sum_{j=1}^{n-m-1}(d_{j+1} + g_m c r_{j+1} - l_{j,(n-m)})x_{j+1} \end{aligned} \tag{51}$$

By choosing

$$l_{j,(n-m)} = d_{j+1} + g_m c r_{j+1} \quad \forall j \in \{1, \ldots, n-m-1\} \tag{52}$$

we get :

$$\dot{X}_{n-m} = g_m(1+\mu q)\sigma(y)u + \omega_{n-m}(y) + g_m c\bar{\xi} + B_1 z \tag{53}$$

where ω_{n-m} is a smooth function, with $\omega_{n-m}(0)=0$, given by :

$$\omega_{n-m}(y) = (d_1 + g_m c r_1)y + \phi_{n-m}(y) - \sum_{j=1}^{n-m-1} l_{j,(n-m)}\phi_j(y) \ . \tag{54}$$

Now suppose that for all $j \in \{i, i+1, \ldots, n-m\}$ where $2 \leq i \leq n-m$, there exist a suitable choice of $l_{k,j}$ $(k \leq j)$ and smooth functions ω_j, $\omega_j(0)=0$, such that :

$$\begin{aligned} \dot{X}_j &= X_{j+1} + \omega_j(y) \ , \quad j \in \{i, \ldots, n-m-1\} & (55) \\ \dot{X}_{n-m} &= g_m(1+\mu q)\sigma(y)u + \omega_{n-m}(y) + g_m c\bar{\xi} + B_1 z & (56) \end{aligned}$$

We will show that (55) holds also for $j = i-1$ after an appropriate choice of the $l_{k,(i-1)}$'s $(k < i-1)$.

From (45) and (37), it follows that :

$$\begin{aligned} \dot{X}_{i-1} &= x_i + \phi_{i-1}(y) - \sum_{j=1}^{i-2} l_{j,(i-1)}(x_{j+1} + \phi_j(y)) \\ &= X_i + l_{1,i}y + \phi_{i-1}(y) \\ &\quad - \sum_{j=1}^{i-2} l_{j,(i-1)}\phi_j(y) + \sum_{j=1}^{i-2}(l_{j+1,i} - l_{j,(i-1)})x_{j+1} \end{aligned} \tag{57}$$

By choosing the $l_{j,(i-1)}$'s as follows :

$$l_{j,(i-1)} = l_{j+1,i} \ , \quad \forall j \in \{1, \ldots, i-2\} \ , \tag{58}$$

(57) implies :

$$\dot{X}_{i-1} = X_i + \omega_{i-1}(y) \tag{59}$$

where ω_{i-1} is smooth, $\omega_{i-1}(0)=0$, defined by :

$$\omega_{i-1}(y) := l_{1,i}y + \phi_{i-1}(y) - \sum_{j=1}^{i-2} l_{j,(i-1)}\phi_j(y) \ . \tag{60}$$

Thus by induction, with appropriate choices of the $l_{j,i}$'s, there exist smooth functions ω_i $(1 \leq i \leq n-m)$, $\omega_i(0)=0$, such that

$$\begin{aligned} \dot{X}_i &= X_{i+1} + \omega_i(y) \ , \quad 1 \leq i \leq n-m-1 \\ \dot{X}_{n-m} &= g_m(1+\mu q)\sigma(y)u + \omega_{n-m}(y) + g_m c\bar{\xi} + B_1 z \end{aligned} \tag{61}$$

With (50) and (61), in the new coordinates $(X_1, \dots, X_{n-m}, z, \bar{\xi})$ defined in (45), the system (37) is transformed into :

$$\begin{array}{rcl} \dot{X}_i &=& X_{i+1} + \omega_i(y) \ , \quad 1 \leq i \leq n-m-1 \\ \dot{X}_{n-m} &=& g_m(1+\mu q)\sigma(y)u + \omega_{n-m}(y) + g_m c\bar{\xi} + B_1 z \\ \dot{z} &=& H_1 z \ + \ G_1(y) \\ \dot{\bar{\xi}} &=& (A - \dfrac{\mu bc}{1+\mu q})\bar{\xi} \ + \ G_2(y) - \dfrac{\mu b}{g_m(1+\mu q)} B_1 z \\ y &=& X_1 \ . \end{array} \tag{62}$$

One sees that the main difference between (62) and (37) is all the dynamic perturbations $(z, \bar{\xi})$ in (62) are now driven by the output y and in the sense of ISS.

It is interesting to note that, when $q = 0$, (62) is nothing but a particular case of the systems considered in [7, eq. (4)]. So in this case, the output feedback stabilization problem of (62) (i.e. (37)) has been solved in [7].

To treat the general case where possibly $q \neq 0$, let us introduce the variables :

$$Y_i = \frac{X_i}{g_m(1+\mu q)} \ , \quad 1 \leq i \leq n-m \ . \tag{63}$$

Then (62) is rewritten as :

$$\begin{array}{rcl} \dot{Y}_i &=& Y_{i+1} + \dfrac{\omega_i(y)}{g_m(1+\mu q)} \ , \quad 1 \leq i \leq n-m-1 \\ \dot{Y}_{n-m} &=& \sigma(y)u + \dfrac{\omega_{n-m}(y) + g_m c\bar{\xi} + B_1 z}{g_m(1+\mu q)} \\ \dot{z} &=& H_1 z \ + \ G_1(y) \\ \dot{\bar{\xi}} &=& (A - \dfrac{\mu bc}{1+\mu q})\bar{\xi} \ + \ G_2(y) - \dfrac{\mu b}{g_m(1+\mu q)} B_1 z \\ y &=& X_1 \ . \end{array} \tag{64}$$

Like in [7], we design the following partial-state observer :

$$\begin{array}{rcl} \dot{\widehat{Y}}_i &=& \widehat{Y}_{i+1} + k_i(y - \widehat{Y}_1) \ , \quad 1 \leq i \leq n-m-1 \\ \dot{\widehat{Y}}_{n-m} &=& \sigma(y)u + k_{n-m}(y - \widehat{Y}_1) \end{array} \tag{65}$$

where the design parameters k_i are chosen in such a way that the observation error $\widetilde{Y} = Y - \widehat{Y}$ (i.e. each component is $\widetilde{Y}_i = Y_i - \widehat{Y}_i$) satisfies :

$$\dot{\widetilde{Y}} = A_0\widetilde{Y} + K\left(1 - \frac{1}{g_m(1+\mu q)}\right)y + \Phi(y, z, \bar{\xi}) \tag{66}$$

with A_0 an asymptotically stable matrix, $K = (k_1, \ldots, k_{n-m})^T$ and Φ the function given by :

$$\Phi(y, z, \bar{\xi}) = \left(\frac{\omega_1(y)}{g_m(1+\mu q)}, \ldots, \frac{\omega_{n-m-1}(y)}{g_m(1+\mu q)}, \frac{\omega_{n-m}(y) + g_m c\bar{\xi} + B_1 z}{g_m(1+\mu q)}\right)^T \tag{67}$$

In view of (63), (64), (65) and (66), we establish the following overall system which is in a position for control design :

$$\begin{aligned}
\dot{y} &= g_m(1+\mu q)\widehat{Y}_2 + g_m(1+\mu q)\widetilde{Y}_2 + \omega_1(y) \\
\dot{\widehat{Y}}_i &= \widehat{Y}_{i+1} + k_i\widetilde{Y}_1\ , \quad 2 \le i \le n-m-1 \\
\dot{\widehat{Y}}_{n-m} &= \sigma(y)u + k_{n-m}(y - \widehat{Y}_1) \\
\dot{\widetilde{Y}} &= A_0\widetilde{Y} + K\left(1 - \frac{1}{g_m(1+\mu q)}\right)y + \Phi(y, z, \bar{\xi}) \\
\dot{z} &= H_1 z \ + \ G_1(y) \\
\dot{\bar{\xi}} &= \left(A - \frac{\mu bc}{1+\mu q}\right)\bar{\xi} \ + \ G_2(y) - \frac{\mu b}{g_m(1+\mu q)}B_1 z
\end{aligned} \tag{68}$$

Note that, from [3, Proposition 3.2] or [1, Proposition 2.4], the composite $(\widetilde{Y}, z, \bar{\xi})$-subsystem of (68) is again an ISS system with respect to y. Let γ be its gain function. It is easy to check that $K(1 - \frac{1}{g_m(1+\mu q)})$ and the unknown multiplicative factors in $\Phi(y, z, \bar{\xi})$, $G_2(y) - \frac{\mu b}{g_m(1+\mu q)}B_1 z$, $\omega_1(y)$ are bounded by $K_0 + K_1|\mu|$ for a *known* positive constant K_0, certain positive constant K_1 and sufficiently small μ. This fact together with the smoothness of the functions in question implies that the function γ is bounded by a known smooth $\mathcal{K}_\infty$-function which is linear near zero when μ is sufficiently small.

Therefore, like for system (32), by applying recursively [3, Corollary 2.2, Proposition 3.2 & Fact 1] to system (68) (also see [3, Section 4], [1, Section 2.2.2]), we arrive at constructing a smooth control law $u = \vartheta(y, \widehat{Y}_2, \ldots, \widehat{Y}_{n-m})$ which globally asymptotically stabilizes (68) and then systems (64), (62).

Finally, summarizing all above, we have :

Theorem 3 *We can find a smooth dynamic output feedback $\vartheta(y, \chi)$, $\dot{\chi} = \varpi(y, \chi)$ so that, for some positive constant μ^* and all μ in $(-\mu^*, \mu^*)$, the null solution of the closed loop system (37) and $\dot{\chi} = \varpi(y, \chi)$ is GAS.*

4 Conclusion

It has been shown in this note that the three classes of nonlinear systems with input unmodeled dynamics in [5] are actually transformed into systems with ISS unmodeled dynamics generated by the state. Therefore, the stabilization results of [5] may be re-obtained by means of the small-gain results in our previous work.

We like to point out that extensions to more general input disturbances of the form $\dot{\xi} = h(\xi, u, x)$ with h *linear* in u will be possible. However, it seems hard to apply the techniques in this note to nonlinear input disturbances. On the other hand, even in the presence of linear unmodeled dynamics acting on the input, it is not uninteresting to compare numerical performance of the controls by Krstić, Sun and Kokotović [5] with ours.

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References

[1] Z. P. Jiang, *Quelques résultats de stabilisation robuste. Applications à la commande*, Thèse de l'Ecole des Mines de Paris, 1993.

[2] Z. P. Jiang and L. Praly, Technical results for the study of robustness of Lagrange stability, *Systems & Control Letters*, 1994, to appear.

[3] Z. P. Jiang, A. Teel and L. Praly, Small-gain theorem for ISS systems and applications, submitted to *Mathematics of Control, Signals and Systems*, revised March 16, 1994.

[4] I. Kanellakopoulos, P.V. Kokotović and A.S. Morse, A toolkit for nonlinear feedback design, *Systems & Control Letters*, **18** (1992), 83-92.

[5] M. Krstić, J. Sun and P. V. Kokotović, Robust control of nonlinear systems with input unmodeled dynamics, *Technical Report CCEC 94-0125*, University of California, 1994.

[6] L. Praly, Commande adaptative indirecte multivariable, *Colloque National du CNRS*, Belle-Ile, 1982.

[7] L. Praly and Z. P. Jiang, Stabilization by output feedback for systems with ISS inverse dynamics, *Systems & Control Letters* **21** (1993) 19-33.

[8] E. D. Sontag, Smooth stabilization implies coprime factorization, *IEEE Trans. Automat. Control*, **34** (1989), 435-443.

[9] A. Teel and L. Praly, Tools for semi-global stabilization by partial state and output feedback, *SIAM J. Control and Optimization*, to appear.

[10] A. Teel and L. Praly, On output-feedback stabilization for systems with ISS inverse dynamics and uncertainties, *Proc. 32nd IEEE Conf. Dec. Control*, pp. 1942-1947, 1993.

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